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# Extension of VIKOR method for decision making problem with interval numbers

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ABSTRACT

The VIKOR method was developed for multi-criteria optimization of complex systems. It determines the compromise ranking list and the compromise solution obtained with the initial (given) weights. This method focuses on ranking and selecting from a set of alternatives in the presence of conflicting criteria. It introduces the multi-criteria ranking index based on the particular measure of "closeness" to the "ideal" solution. The aim of this paper is to extend the VIKOR method for decision making problems with interval number. The extended VIKOR method's ranking is obtained through comparison of interval numbers and for doing the comparisons between intervals, we introduce $\alpha$ as optimism level of decision maker. Finally, a numerical example illustrates and clarifies the main results developed in this paper.



## 1. Introduction

Multi-criteria optimization is the process of determining the best feasible solution according to the established criteria (representing different effects). Practical problems are often characterized by several non-commensurable and conflicting criteria and there may be no solution satisfying all criteria simultaneously. Thus, the solution is a set of non-inferior solutions, or a compromise solution according to the decision maker's preferences. The compromise solution was established by Yu [1] and Zeleny [2] for a problem with conflicting criteria and it can be helping the decision makers to reach a final solution. The compromise solution is a feasible solution, which is the closest to the ideal, and compromise means an agreement established by mutual concessions.

A multi attribute decision making (MADM) problem can be defined as:

| | $C_1$ | $C_2$ | $\dots$ | $C_n$ |
|---|---|---|---|---|
| $A_1$ | $f_{11}$ | $f_{12}$ | $\dots$ | $f_{1n}$ |
| $A_2$ | $f_{21}$ | $f_{22}$ | $\dots$ | $f_{2n}$ |
| $\dots$ | $\dots$ | $\dots$ | $\dots$ | $\dots$ |
| $A_m$ | $f_{m1}$ | $f_{m2}$ | $\dots$ | $f_{mn}$ |

$$W = [w_1, w_2, \dots, w_n]$$

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where $A_1, A_2, \ldots, A_m$ are possible alternatives among which decision makers have to choose, $C_1, C_2, \ldots, C_n$ are criteria with which alternative performance is measured, $f_{ij}$ is the rating of alternative $A_i$ with respect to criterion $C_j$, $w_j$ is the weight of criterion $C_j$ [3–5].

In classical MCDM methods, the ratings and the weights of the criteria are known precisely, whereas in the real world, in an imprecise and uncertain environment, it is an unrealistic assumption that the knowledge and representation of a decision maker or expert are so precise. For example, human judgment including preferences is often vague and decision maker (DM) cannot estimate his preference with exact numerical values. In these situations, determining the exact value of the attributes is difficult or impossible. So, to describe and treat imprecise and uncertain elements present in a decision problem, fuzzy and stochastic approaches are frequently used. In the literature, in the works of fuzzy decision making [6–8], fuzzy parameters are assumed to be with known membership functions and in stochastic decision making [9–12] parameters are assumed to have known probability distributions. However, in reality to a decision maker (DM) it is not always easy to specify the membership function or probability distribution in an inexact environment. At least in some of the cases, the use of interval numbers may serve the purpose better. An interval number can be thought as an extension of the concept of a real number and also as a subset of the real line $\Re$ [13]. However, in decision problems its use is not much attended as it merits.

Recently, Jahanshahloo et al. [14] have extended TOPSIS method to solve decision making problems with interval data. According to a comparative analysis of VIKOR and TOPSIS written by Opricovic and Tzeng [15], VIKOR method and TOPSIS method use different aggregation functions and different normalization methods. TOPSIS method is based on the principle that the optimal point should have the shortest distance from the positive ideal solution (PIS) and the farthest from the negative ideal solution (NIS). Therefore, this method is suitable for cautious (risk avoider) decision maker(s), because the decision maker(s) might like to have a decision which not only makes as much profit as possible, but also avoids as much risk as possible. Besides, computing the optimal point in the VIKOR is based on the particular measure of "closeness" to the PIS. Therefore, it is suitable for those situations in which the decision maker wants to have maximum profit and the risk of the decisions is less important for him. Therefore, we extend the concept of VIKOR method to develop a methodology for solving MADM problems with interval numbers.

The VIKOR method is presented in the next section. In section 3, extended VIKOR method is introduced and a new method is proposed for interval ranking on the basis of decision maker's optimistic level. In Section 4, an illustrative example is presented to show an application of extended VIKOR method. Finally, conclusion is presented.

## 2. VIKOR method

The VIKOR method was introduced as one applicable technique to be implemented within MCDM problem and it was developed as a multi attribute decision making method to solve a discrete decision making problem with non-commensurable (different units) and conflicting criteria [15,16]. This method focuses on ranking and selecting from a set of alternatives, and determines compromise solution for a problem with conflicting criteria, which can help the decision makers to reach a final solution. The multi-criteria measure for compromise ranking is developed from the $L_P$-*metric* used as an aggregating function in a compromise programming method [1,2].

Assuming that each alternative is evaluated according to each criterion function, the compromise ranking could be performed by comparing the measure of closeness to the ideal alternative. The various $m$ alternatives are denoted as $A_1$, $A_2, \ldots, A_m$. For alternative $A_i$, the rating of the $j$th aspect is denoted by $f_{ij}$, i.e. $f_{ij}$ is the value of $j$th criterion function for the alternative $A_i$; $n$ is the number of criteria. Development of the VIKOR method is started with the following form of $L_P$-*metric*:

$$L_{pi} = \left\{ \sum_{j=1}^{n} [(f_j^* - f_{ij})/(f_j^* - f_j^-)]^p \right\}^{1/p} \quad 1 \leqslant p \leqslant \infty; \quad i = 1, 2, \ldots, m. \tag{1}$$

In the VIKOR method $L_{1,i}$ (as $S_i$) and $L_{\infty,i}$ (as $R_i$) are used to formulate ranking measure. The solution obtained by min $S_i$ is with a maximum group utility ("majority" rule), and the solution obtained by min $R_i$ is with a minimum individual regret of the "opponent".

The compromise ranking algorithm of the VIKOR method has the following steps:

(a) Determine the best $f_j^*$ and the worst $f_j^-$ values of all criterion functions $j = 1,2,\ldots,n$. If the $j$th function represents a benefit then:

$$f_j^* = \max_i f_{ij}, \quad f_j^- = \min_i f_{ij} \tag{2}$$

(b) Compute the values $S_i$ and $R_i$; $i = 1,2,\ldots,m$, by these relations:

$$S_i = \sum_{j=1}^{n} w_j (f_j^* - f_{ij})/(f_j^* - f_j^-), \tag{3}$$

$$R_i = \max_j w_j (f_j^* - f_{ij})/(f_j^* - f_j^-), \tag{4}$$

where $w_j$ are the weights of criteria, expressing their relative importance.

(c) Compute the values $Q_i$; $i = 1,2,\ldots,m$, by the following relation:

$$Q_i = v(S_i - S^*)/(S^- - S^*) + (1 - v)(R_i - R^*)/(R^- - R^*) \tag{5}$$

where

$$S^* = \min_i S_i, \quad S^- = \max_i S_i, \tag{6}$$
$$R^* = \min_i R_i, \quad R^- = \max_i R_i, \tag{7}$$

$v$ is introduced as weight of the strategy of "the majority of criteria" (or "the maximum group utility"), here suppose that $v = 0.5$.

(d) Rank the alternatives, sorting by the values $S$, $R$ and $Q$ in decreasing order. The results are three ranking lists.

(e) Propose as a compromise solution the alternative $A'$, which is ranked the best by the measure $Q$ (Minimum) if the following two conditions are satisfied:

**C1**. *Acceptable advantage*:

$$Q(A'') - Q(A') \geqslant DQ$$

where $A''$ is the alternative with second position in the ranking list by $Q$; $DQ = 1/(m - 1)$; $m$ is the number of alternatives.

**C2**. *Acceptable stability in decision making*:

Alternative $A'$ must also be the best ranked by $S$ or/and $R$. This compromise solution is stable within a decision making process, which could be "voting by majority rule" (when $v > 0.5$ is needed), or "by consensus" $v \approx 0.5$, or "with veto" ($v < 0.5$). Here, $v$ is the weight of the decision making strategy "the majority of criteria" (or "the maximum group utility").

If one of the conditions is not satisfied, then a set of compromise solutions is proposed, which consists of:

- Alternatives $A'$ and $A''$ if only condition C2 is not satisfied, or
- Alternatives $A', A'', \ldots, A^{(M)}$ if condition C1 is not satisfied; $A^{(M)}$ is determined by the relation $Q(A^{(M)}) - Q(A') < DQ$ for maximum $M$ (the positions of these alternatives are "in closeness").

The best alternative, ranked by $Q$, is the one with the minimum value of $Q$. The main ranking result is the compromise ranking list of alternatives, and the compromise solution with the "advantage rate". VIKOR is an effective tool in multi-criteria decision making, particularly in a situation where the decision maker is not able, or does not know to express his/her preference at the beginning of system design. The obtained compromise solution could be accepted by the decision makers because it provides a maximum "group utility" (represented by min $S$) of the "majority", and a minimum of the "individual regret" (represented by min $R$) of the "opponent". The compromise solutions could be the basis for negotiations, involving the decision maker's preference by criteria weights.

## 3. Extended VIKOR method for decision making problem with interval numbers

As it was said in the introduction, the interval numbers are more suitable to deal with the decision making problems in the imprecise and uncertain environment, because they are the simplest form of representing uncertainty in the decision matrix. The interval numbers require the minimum amount of information about the values of attributes. Specifying an interval for a parameter in decision matrix indicates that the parameter can take any value within the interval. Note that, the interval numbers does not indicate how probable it is to the value to be in the interval, nor does it indicate which of the many values in the interval is the most likely to occur [17]. In other way, an interval number can be thought as:

(1) An extension of the concept of a real number and also as a subset of the real line.
(2) A degenerate flat fuzzy number or fuzzy interval with zero left and right spreads.
(3) An $\alpha$-cut of a fuzzy number [18].

So an interval number signifies the extent of tolerance or a region that the parameter can possibly take. An extensive research and wide coverage on interval arithmetic and its applications can be found in [13,19,20]. More information about the interval numbers and its differences with other methods of representing uncertainty such as probability and fuzzy theory can be found in [18,21,22].

According to these facts, when determining the exact values of the attributes is difficult or impossible, it is more appropriate to consider them as interval numbers. Therefore, in the present paper, we extend the VIKOR method to solve MADM problem with interval numbers. To do this, suppose that a decision matrix with interval numbers has the following form:

| | $C_1$ | $C_2$ | ... | $C_n$ |
|---|---|---|---|---|
| $A_1$ | $[f_{11}^L, f_{11}^U]$ | $[f_{12}^L, f_{12}^U]$ | ... | $[f_{1n}^L, f_{1n}^U]$ |
| $A_2$ | $[f_{21}^L, f_{21}^U]$ | $[f_{22}^L, f_{22}^U]$ | ... | $[f_{2n}^L, f_{2n}^U]$ |
| ... | ... | ... | ... | ... |
| $A_m$ | $[f_{m1}^L, f_{m1}^U]$ | $[f_{m2}^L, f_{m2}^U]$ | ... | $[f_{mn}^L, f_{mn}^U]$ |

$$W = [w_1, w_2, \ldots, w_n]$$

where $A_1, A_2, \ldots, A_m$ are possible alternatives among which decision makers have to choose, $C_1, C_2, \ldots, C_n$ are criteria with which alternative performance are measured, $f_{ij}$ is the rating of alternative $A_i$ with respect to criterion $C_j$ and is not known exactly and only we know $f_{ij} \in \left[f_{ij}^L, f_{ij}^U\right]$ and $w_j$ is the weight of criterion $C_j$. The extended VIKOR method consists of the following steps:

(a) Determine the PIS and NIS.

$$A^* = \{f_1^*, \ldots, f_n^*\} = \left\{\left(\max_i f_{ij}^U \mid j \in I\right) \text{ or } \left(\min_i f_{ij}^L \mid j \in J\right)\right\} \quad j = 1, 2, \ldots, n \tag{8a}$$

$$A^- = \{f_1^-, \ldots, f_n^-\} = \left\{\left(\min_i f_{ij}^L \mid j \in I\right) \text{ or } \left(\max_i f_{ij}^U \mid j \in J\right)\right\} \quad j = 1, 2, \ldots, n \tag{8b}$$

where $I$ is associated with benefit criteria, and $J$ is associated with cost criteria. $A^*$ and $A^-$ are PIS and NIS.

(b) In this step, compute $\left[S_i^L, S_i^U\right]$ and $\left[R_i^L, R_i^U\right]$ intervals as below:

$$S_i^L = \sum_{j \in I} w_j \left(\frac{f_j^* - f_{ij}^U}{f_j^* - f_j^-}\right) + \sum_{j \in J} w_j \left(\frac{f_{ij}^L - f_j^*}{f_j^- - f_j^*}\right) \quad i = 1, \ldots, m \tag{9a}$$

$$S_i^U = \sum_{j \in I} w_j \left(\frac{f_j^* - f_{ij}^L}{f_j^* - f_j^-}\right) + \sum_{j \in J} w_j \left(\frac{f_{ij}^U - f_j^*}{f_j^- - f_j^*}\right) \quad i = 1, \ldots, m \tag{9b}$$

$$R_i^L = \max\left\{ w_j \left(\frac{f_j^* - f_{ij}^U}{f_j^* - f_j^-}\right) \middle| j \in I, \quad w_j \left(\frac{f_{ij}^L - f_j^*}{f_j^- - f_j^*}\right) \middle| j \in J \right\} \quad i = 1, \ldots, m \tag{10a}$$

$$R_i^U = \max\left\{ w_j \left(\frac{f_j^* - f_{ij}^L}{f_j^* - f_j^-}\right) \middle| j \in I, \quad w_j \left(\frac{f_{ij}^U - f_j^*}{f_j^- - f_j^*}\right) \middle| j \in J \right\} \quad i = 1, \ldots, m \tag{10b}$$

(c) Compute the interval $Q_i = [Q_i^L, Q_i^U]$; $i = 1,2,\ldots,m$, by these relations:

$$Q_i^L = v \frac{(S_i^L - S^*)}{(S^- - S^*)} + (1 - v) \frac{(R_i^L - R^*)}{(R^- - R^*)} \tag{11a}$$

$$Q_i^U = v \frac{(S_i^U - S^*)}{(S^- - S^*)} + (1 - v) \frac{(R_i^U - R^*)}{(R^- - R^*)} \tag{11b}$$

where

$$S^* = \min_i S_i^L, \quad S^- = \max_i S_i^U, \tag{12}$$

$$R^* = \min_i R_i^L, \quad R^- = \max_i R_i^U, \tag{13}$$

$v$ is introduced as weight of the strategy of "the majority of criteria" (or "the maximum group utility"), here suppose that, $v = 0.5$.

(d) Based on the VIKOR method, the alternative that has minimum $Q_i$ is the best alternative and it is chosen as compromise solution. But here the $Q_i$, $i = 1, \ldots, m$ *are interval numbers*. To choose the minimum *interval number* they are compared with each other. So, we introduce a new method for comparison of interval numbers as follows:

(e) Suppose that $[a^L, a^U]$ and $[b^L, b^U]$, are two interval numbers that we want to choose minimum interval number between them. These two interval numbers can have four status:

(1) If these interval numbers have no intersection, the minimum interval number is the one that has lower values. In other words: If $a^U \leqslant b^L$ then, we choose $[a^L, a^U]$ as minimum interval number.

(2) If two interval numbers are the same, both of them have the same priority for us.

(3) In situations that $a^L \leqslant b^L < b^U \leqslant a^U$, we choose minimum interval number in this way: If $\alpha(b^L - a^L) \geqslant (1 - \alpha)(a^U - b^U)$ then $[a^L, a^U]$ is our minimum interval number, else $[b^L, b^U]$ is minimum interval number.

(4) In situations that $a^L < b^L < a^U < b^U$, if $\alpha\,(b^L - a^L) \geqslant (1-\alpha)(b^U - a^U)$, then $[a^L, a^U]$ is minimum interval number, else $[b^L, b^U]$ is minimum interval number.

Here, $\alpha$ is introduced as optimism level of the decision maker ($0 < \alpha \leqslant 1$). The optimist decision maker has greater $\alpha$ value than the pessimist decision maker. For rational decision maker $\alpha = 0.5$ and in this situation the result of the comparisons obtained by the introduced method is similar to interval numbers comparison that has been made on the basis of interval numbers means.

## 4. Numerical example

In this section, we present a numerical example to illustrate how the proposed method can be used. Suppose that, there are three alternatives ($A_1$, $A_2$, $A_3$) *and two criteria* ($C_1$, $C_2$). The decision maker wants to choose an alternative that has minimum $C_1$ and maximum $C_2$. The values of decision matrix are not precise and interval numbers are used to describe and treat the uncertainty of the decision problem. The interval decision matrix is shown in Table 1.

In this example, both criteria have similar relative importance, $w_j = 0.5$, $j = 1,2$. The decision maker claims his optimism level is $\alpha = 0.6$. Now let's suppose that, $v = 0.5$.

To solve this example using the extended VIKOR method we go through the following steps.

(a) The PIS and NIS are computed by (8a) and (8b) and shown in Table 2.

(b) In this step, we compute $\left[S_i^L, S_i^U\right]$ and $\left[R_i^L, R_i^U\right]$ using (9a), (9b), (10a) and (10b). The result is presented in Table 3.

(c) We compute the interval $Q_i = \left[Q_i^L, Q_i^U\right]$; $i = 1, 2, \ldots, m$, by (11a), (11b), (12) and (13). The results are shown in Table 4.

$$S^* = 0.3341 \quad S^- = 0.5717$$
$$R^* = 0.2172 \quad R^- = 0.5000.$$

(d) Using (d) and (e) in Section 3, final ranking is obtained as follows:

$$\begin{cases} A_1 > A_3 \\ A_2 > A_1 \\ A_2 > A_3 \end{cases} \Rightarrow \text{Final ranking is} : A_2 > A_1 > A_3.$$

**Table 1**
Interval decision matrix

| | $C_1$ | $C_2$ |
|---|---|---|
| $A_1$ | [0.75, 1.24] | [2784, 3192] |
| $A_2$ | [1.83, 2.11] | [3671, 3857] |
| $A_3$ | [4.90, 5.73] | [4409, 4681] |

**Table 2**
PIS and NIS

| | $C_1$ | $C_2$ |
|---|---|---|
| $f_j^*$ | 0.75 | 4681 |
| $f_j^-$ | 5.37 | 2784 |

**Table 3**
$S$ and $R$ interval numbers

| | $\left[S_i^L, S_i^U\right]$ | $\left[R_i^L, R_i^U\right]$ |
|---|---|---|
| $A_1$ | [0.3925, 0.5530] | [0.3925, 0.5000] |
| $A_2$ | [0.3341, 0.4134] | [0.2172, 0.2662] |
| $A_3$ | [0.4491, 0.5717] | [0.4491, 0.5000] |

**Table 4**
$Q$ interval numbers

| | $\left[Q_i^L, Q_i^U\right]$ |
|---|---|
| $A_1$ | [0.4328, 0.9606] |
| $A_2$ | [0.0000, 0.2535] |
| $A_3$ | [0.6520, 1.0000] |

The compromise solution of extended VIKOR method is $A_2$.

As mentioned in the introduction, Jahanshahloo et al. [9] have extended TOPSIS method to solve decision making problems with interval data. This method uses different aggregation functions and different normalization methods. Here to make a comparison between these two methods, we solve this example using the extended TOPSIS method.

Doing the introduced steps in the extended TOPSIS method, compromise solution is obtained as follows:

$$\text{The ranking of extended TOPSIS is}: A_1 > A_2 > A_3.$$

The compromise solution obtained by extended TOPSIS is different with the compromise solution of extended VIKOR. These different solutions derive from differences in aggregation functions and normalization methods. Moreover, in extended TOPSIS, the interval numbers are reduced to exact values. These reductions lead to miss some information. In the extended VIKOR method by keeping interval numbers, considering the decision maker's optimism level and using the comparison of interval numbers, the compromise solution is obtained.

## 5. Conclusion

Because of the fact that determining the exact values of the attributes is difficult or impossible, it is more appropriate to consider them as interval numbers. In this paper, we extended the VIKOR method to MADM problem with interval numbers. This method introduced the ranking index based on particular measure of closeness to PIS. In the extended VIKOR method, we compute $S$, $R$ *and* $Q$ as interval numbers and to obtain the compromise solution, we need to compare interval numbers with each other. To do this, we have introduced new method on the basis of $\alpha$ as optimism level of the decision maker. The ranking result in this method depends on $\alpha$.

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