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## A Concentration Ratio for Nonlinear Best Worst Method

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Best Worst Method (BWM) is a multi-criteria decision-making method that is based on a structured pairwise comparison system. It uses two pairwise comparison vectors (*best-to-others* and *others-to-worst*) as input for an optimization model to get the optimal weights of the criteria (or alternatives). The original BWM involves a nonlinear model that sometimes results in multiple optimal weights meaning that the weight of each criterion is presented as an interval. The aim of this paper is to introduce a ratio, called concentration ratio, to check the concentration of the optimal intervals obtained from the nonlinear BWM. The relationship between the concentration ratio and the consistency ratio is investigated and it is found that the concentration ratio along with the consistency ratio of the model provides enhanced insights into the reliability and flexibility of the results of BWM.

*Keywords:* Multi-criteria decision-making; best worst method (BWM); concentration; consistency.

### 1. Introduction

Multi-criteria decision-making (MCDM) is a growing field of research that assists decision-makers (DMs) in identifying the best alternative from a set of alternatives  $A = \{a_1, a_2, \dots, a_m\}$  that are evaluated with respect to a set of decision-making criteria  $C = \{c_1, c_2, \dots, c_n\}$ . A normalized performance matrix  $P = \{p_{ij}\}$  represents the normalized scores of alternative  $i$  with respect to criterion  $j$ . Considering the importance (weight) of the decision-making criteria as  $w = \{w_1, w_2, \dots, w_n\}$ , the overall value of each alternative  $i$ ,  $v_i$ , can be obtained using different approaches such as the following additive value function<sup>1</sup>:

$$v_i = \sum_{j=1}^n w_j p_{ij}. \quad (1)$$

An imperative part of this problem is the provision of the importance (weight) of the criteria,  $w_j$ . There are several methods developed to identify these weights,

including Trade-off weighting,<sup>1</sup> Swing weighting,<sup>2</sup> SMART, SMARTS and SMARTER (simple multi-attribute rating technique),<sup>3</sup> AHP (analytic hierarchy process)<sup>4</sup> and BWM (best worst method).<sup>5</sup> For more information about the MCDM methods, see Triantaphyllou.<sup>6</sup>

The focus of this paper is on BWM, which has been applied to a wide range of application areas, including the airline industry,<sup>7-9</sup> supplier selection,<sup>10-13</sup> technology assessment and selection,<sup>14-16</sup> location selection,<sup>17</sup> quality assessment of scientific outputs,<sup>18</sup> and energy<sup>19,20</sup> among others. To see more applications and extensions of BWM, we refer to the review paper Mi *et al.*<sup>21</sup>

BWM is a pairwise comparison-based method that offers a structured way to make the comparisons. This structure brings several major benefits: (i) By identifying the best and the worst criteria (or the alternatives) before conducting the pairwise comparisons among the criteria (or the alternatives), the DM already has a clear understanding of the range of evaluation which could lead to more reliable pairwise comparisons. This, in turn, implies more consistent pairwise comparisons, which has been shown in the original study of Rezaei.<sup>5</sup> (ii) The use of two pairwise comparisons vectors formed based on two opposite references (best and worst) in a single optimization model could mitigate possible anchoring bias that the DM might have during the process of conducting pairwise comparisons. This so-called consider-the-opposite-strategy has been shown to be an effective strategy in mitigating the anchoring bias in other studies.<sup>22</sup> (iii) In pairwise comparison-based methods we either have methods for which we use a single vector (e.g., Swing and SMART family) or a full matrix (e.g., AHP). Although using one vector for the input data makes the method very data (and time)-efficient, the main weakness of methods based on only one vector is that the consistency of the provided pairwise comparisons cannot be checked. On the other hand, although using a full matrix provides the possibility of checking the consistency of the provided pairwise comparisons, methods which are based on full pairwise comparison matrix are not data (and time)-efficient. Asking too many questions from the DM, which occurs in the case of full matrix, might even contribute to the confusion and inconsistency of the DM. BWM stands in the middle. That is to say, it is the most data (and time)-efficient method which could, at the same time, provide the possibility of checking the consistency of the provided pairwise comparisons. As the two vectors are formed with considering two specific reference criteria (or alternatives), BWM should not be seen as a case of incomplete pairwise comparison matrix.<sup>23</sup> (iv) BWM (its original nonlinear model), in the not-fully-consistent cases with more than three criteria (or alternatives) might bring about multiple optimal solutions. This is a reflection of the inconsistency which exists in the provided data. Having multiple optimal solutions (compared to a unique solution) brings more flexibility to the cases where there are multiple DMs involved. This means that in the context of group decision-making, having multiple optimal solutions (for all or some DMs) could result in a higher chance (compared to the case that each DM has a unique solution) for a compromise

solution to coincide (or at least be very close) to one of the optimal solutions. Although having multiple optimal weights is advantageous in some cases, especially in group decision-making problems,<sup>24–28</sup> where debating plays a central role,<sup>29</sup> in other cases, having a unique solution is preferred. Rezaei<sup>30</sup> developed a linear approximation model for the BWM which provides a unique solution.

The reliability of the weights obtained from the method is checked using a consistency ratio, which is calculated after solving the optimization model.<sup>5</sup> Since the nonlinear model, sometimes, provides more than one optimal solution, it is also important to check the concentration of the results, which is the main aim of this study. The proposed ratio shows the relative wideness of the optimal intervals which allows different interpretations. For instance, it shows the level of uncertainty of the DM in providing the pairwise comparisons which is reflected in the optimal intervals. It can also be seen, especially in the case of group decision-making, as the level of freedom for the group members. That is to say, the wider the intervals, the more freedom the group members have over choosing a compromise solution. The relationship between this new ratio and the consistency ratio provides very interesting insights which are discussed in the next sections.

The remainder of the paper is organized as follows. In Sec. 2, a brief overview of BWM is presented. In Sec. 3, a ratio is proposed to check the concentration of the results of the nonlinear BWM. Finally, the conclusion is provided in Sec. 4.

## 2. A Brief Overview of the Nonlinear BWM

First, the DM identifies a set of decision-making criteria,  $C = \{c_1, c_2, \dots, c_n\}$ , which contributes to the goal of the decision-making problem. The DM then identifies the *best* (B) (e.g., most important, most desirable, most contributing) and the *worst* (W) (e.g., least important, least desirable, least contributing) decision-making criteria. The DM expresses his/her preferences regarding the best over the other criteria as  $a_{Bj}$  and of all the criteria over the worst as  $a_{jW}$ , using a number between 1 to 9 (1 means criterion  $i$  is equally important to criterion  $j$ , while 9 means criterion  $i$  is extremely more important than criterion  $j$ ). In order to obtain the most consistent weights with the pairwise comparisons, the maximum distance between the pairwise comparisons and their corresponding weight ratios should be minimized, or equivalently:

$$\begin{aligned} & \min \max_j \left\{ \left| \frac{w_B}{w_j} - a_{Bj} \right|, \left| \frac{w_j}{w_W} - a_{jW} \right| \right\}, \\ & \text{s.t.} \\ & \sum_{j=1}^n w_j = 1, \\ & w_j \geq 0, \quad \forall j. \end{aligned} \tag{2}$$

Model (2) is converted to the following model:

$$\begin{aligned}
 &\min \xi, \\
 &\text{s.t.} \\
 &\left| \frac{w_B}{w_j} - a_{Bj} \right| \leq \xi, \quad \forall j, \\
 &\left| \frac{w_j}{w_W} - a_{jW} \right| \leq \xi, \quad \forall j, \\
 &\sum_{j=1}^n w_j = 1, \\
 &w_j \geq 0, \quad \forall j.
 \end{aligned} \tag{3}$$

Solving model (3), the optimal weights  $(w_1^*, w_2^*, \dots, w_n^*)$  are obtained.

The consistency ratio of the model is calculated using the following formula:

$$\text{Consistency Ratio} = \frac{\xi^*}{\text{CI}}, \tag{4}$$

where  $\xi^*$  is the optimal objective value of model (3), and CI is the consistency index which can be read from Table 1.

The consistency ratio represents the veracity between the obtained weights and the pairwise comparison data provided by the DM. If the consistency ratio is not greater than a fixed threshold,<sup>31</sup> the results are acceptable, otherwise the provided pairwise comparisons need to be revised (see Table 2 for the CR thresholds).

If the number of criteria is more than three and the comparison system is not fully consistent, model (3) may provide multiple optimal solutions. In other words, instead of a unique optimal weight for each criterion, we have an optimal interval.

In order to determine the minimum and maximum optimal weights of the criteria (the lower and upper bounds of the intervals), the following two linear programming (LP) problems should be formulated and solved for each criterion.<sup>30</sup>

$$\begin{aligned}
 &\min w_j, \\
 &\text{s.t.} \\
 &|w_B - a_{Bj}w_j| \leq \xi^*w_j, \quad \forall j, \\
 &|w_j - a_{jW}w_W| \leq \xi^*w_W, \quad \forall j, \\
 &\sum_{j=1}^n w_j = 1, \\
 &w_j \geq 0, \quad \forall j,
 \end{aligned} \tag{5}$$

Table 1. CI table.<sup>5</sup>

$a_{BW}$	1	2	3	4	5	6	7	8	9
Consistency Index (max $\xi$ )	0.00	0.44	1.00	1.63	2.30	3.00	3.73	4.47	5.23

Table 2. Consistency ratio (CR) thresholds for different “maximum value used in the pairwise comparison system”,  $a_{BW}$  and number of criteria,  $n$ .<sup>31</sup>

$a_{BW}$	Number of criteria, $n$						
	3	4	5	6	7	8	9
3	0.2087	0.2087	0.2087	0.2087	0.2087	0.2087	0.2087
4	0.1581	0.2352	0.2738	0.2928	0.3102	0.3154	0.3273
5	0.2111	0.2848	0.3019	0.3309	0.3479	0.3611	0.3741
6	0.2164	0.2922	0.3565	0.3924	0.4061	0.4168	0.4225
7	0.2090	0.3313	0.3734	0.3931	0.4035	0.4108	0.4298
8	0.2267	0.3409	0.4029	0.4230	0.4379	0.4543	0.4599
9	0.2122	0.3653	0.4055	0.4225	0.4445	0.4587	0.4747

Note: For the cases of  $a_{BW} = 1, 2$ , and  $n = 2$ , the threshold is zero.

$$\begin{aligned}
 & \max w_j, \\
 & \text{s.t.} \\
 & |w_B - a_{Bj}w_j| \leq \xi^*w_j, \quad \forall j, \\
 & |w_j - a_{jW}w_W| \leq \xi^*w_W, \quad \forall j, \\
 & \sum_{j=1}^n w_j = 1, \\
 & w_j \geq 0, \quad \forall j,
 \end{aligned} \tag{6}$$

where  $\xi^*$  is the optimal objective value of model (3).

After determining the optimal weight intervals  $[w_j^{\min*}, w_j^{\max*}]$ , the DM may pick up an optimal weight from the optimal interval, based on some higher-level information or, alternatively, calculate the centre of the interval as a representative optimal weight (see, for instance, Rezaei *et al.*<sup>32</sup>), as follows:

$$w_j^* = \frac{(w_j^{\min*} + w_j^{\max*})}{2}, \tag{7}$$

where  $w_j^{\min*}$  and  $w_j^{\max*}$  are the optimal results found from models (5) and (6), respectively.

### 3. Concentration Ratio for the Nonlinear BWM

As mentioned before, CR measures the level of veracity between the final weight ratios and their corresponding comparisons.  $CR = 0$  means a perfect veracity between the final weight ratios and their corresponding pairwise comparisons, which is equivalent to a unique set of weights. As CR is getting distant from 0, it shows that the pairwise comparison system is not fully consistent. This inconsistency provides some degree of flexibility in the final results. That is to say, the formulation of the nonlinear BWM, given some level of inconsistency, allows for multiple optimal solutions. To determine the lower and upper bounds of the optimal weights of the criteria, we solve models (5) and (6), respectively. While having some level of

flexibility could provide freedom for the members of a group over choosing a compromise solution in a group decision-making, having too much flexibility in the final results, i.e. having too wide ranges for the optimal weight intervals might not be advantageous for a DM. This is why, an important question arises: *To what extent the final results are concentrated?*

In what follows, a concentration ratio is developed to answer this question. The word *concentration* used in this study has conceptual similarity to ‘concentration parameter’ used in statistical analysis.<sup>33,34</sup>

To test the concentration of the results, the following concentration ratio ( $\kappa$ ) is proposed.

**Definition 1.** Concentration ratio of a BWM problem with  $n > 3$  is obtained by  $\kappa = 1 - \frac{\psi}{\max_{a_{BW},n}(\psi)}$ , where  $\psi = \sum_{j=1}^n (w_j^{\max*} - w_j^{\min*})$ , for all  $j$ :  $w_j^{\min*}$  and  $w_j^{\max*}$  are the lower and upper bounds of weight of criterion  $j$ , and  $\max_{a_{BW},n}(\psi)$  is the maximum possible sum of the optimal interval ranges of the weights for a problem with the same  $a_{BW}$  and  $n$ .

While  $\psi$  is calculated after solving a particular problem,  $\max_{a_{BW},n}(\psi)$  are fixed values which will be found in this section. As discussed before for BWM problems with less than 4 criteria, there is always a unique solution and as a result the concentration ratio is 1.

Here, we show how  $\max_{a_{BW},n}(\psi)$  can be found for different values of  $a_{BW}$  and  $n > 3$ .

**Proposition 1.**  $\max_{a_{BW},n}(\psi)$  for a problem characterized with  $a_{BW}$  and  $n$  is realized when, for all  $j$ ,  $\xi = \xi_{\max}$ ,  $a_{Bj} = \lfloor \xi_{\max} \rfloor + 1$ , and  $a_{jW} = \lfloor \xi_{\max} \rfloor + 1$  or  $a_{jW} = \lfloor \xi_{\max} \rfloor + 2$ .

**Proof.** Consider model (3) again, where for criterion  $j$ , we have two constraints as follows:

$$\left| \frac{w_B}{w_j} - a_{Bj} \right| \leq \xi, \quad \forall j, \tag{8}$$

$$\left| \frac{w_j}{w_W} - a_{jW} \right| \leq \xi, \quad \forall j. \tag{9}$$

From (8), we have

$$\frac{w_B}{a_{Bj} + \xi} \leq w_j \leq \frac{w_B}{a_{Bj} - \xi}, \quad \forall j. \tag{10}$$

And, from (9), we have

$$w_W(a_{jW} - \xi) \leq w_j \leq w_W(a_{jW} + \xi), \quad \forall j. \tag{11}$$

From model (3), we also conclude that, for all  $j$ ,  $w_W \leq w_j \leq w_B$  and  $\sum_{j=1}^n w_j = 1$ .

(10) and (11) are used to find the lower and upper bounds of the weights, hence defining the range of the interval.



In (10), the left-hand side is minimized when we have the maximum value for  $a_{Bj}$  and maximum  $\xi$  which is  $\xi_{\max}$ . The right-hand side is maximized for any  $\xi$ , as long as we have  $a_{Bj} = \lfloor \xi \rfloor + 1$ . Obviously  $a_{Bj} \not\leq \lfloor \xi \rfloor + 1$ , as otherwise the right-hand side becomes negative which contradicts  $w_j \geq 0$ , and  $a_{Bj} > \lfloor \xi \rfloor + 1$  makes the other boundary limit  $w_j \leq w_B$  redundant. Considering both sides of (10), the best value for  $\xi$  would be  $\xi_{\max}$ , and having  $\xi = \xi_{\max}$ , the best value for  $a_{Bj}$  would be  $a_{Bj} \geq \lfloor \xi_{\max} \rfloor + 1$ .

It is evident that for  $a_{Bj} \geq \lfloor \xi_{\max} \rfloor + 1$ , the slop of  $\frac{w_B}{a_{Bj}-\xi}$  is steeper than that of  $\frac{w_B}{a_{Bj}+\xi}$ , which implies that if we move from  $\lfloor \xi_{\max} \rfloor + 1$  to  $\lfloor \xi_{\max} \rfloor + 1 + g$  (for  $g = 1, \dots, a_{BW} - (\lfloor \xi_{\max} \rfloor + 1)$ ) the decrease in  $\frac{w_B}{a_{Bj}-\xi}$  is more than the decrease in  $\frac{w_B}{a_{Bj}+\xi}$ . This is not, however, sufficient to conclude that  $a_{Bj} = \lfloor \xi_{\max} \rfloor + 1$  generates the widest range for  $w_j$ . This is because as soon as we move from  $a_{Bj} = \lfloor \xi_{\max} \rfloor + 1$  to  $a_{Bj} = \lfloor \xi_{\max} \rfloor + 2$ , the upper bound  $w_B$  i.e.,  $w_j \leq w_B$  becomes redundant and although the decrease in the upper bound  $\frac{w_B}{a_{Bj}-\xi}$  is  $\frac{w_B}{\lfloor \xi_{\max} \rfloor + 1 - \xi} - \frac{w_B}{\lfloor \xi_{\max} \rfloor + 2 - \xi}$ , the effective decrease is from  $w_B$  to  $\frac{w_B}{\lfloor \xi_{\max} \rfloor + 2 - \xi}$  which is  $\Delta_1 = w_B - \frac{w_B}{\lfloor \xi_{\max} \rfloor + 2 - \xi}$ . The decrease in the lower bound  $\frac{w_B}{a_{Bj}+\xi}$  due to moving from  $a_{Bj} = \lfloor \xi_{\max} \rfloor + 1$  to  $\lfloor \xi_{\max} \rfloor + 2$  is  $\Delta_2 = \frac{w_B}{\lfloor \xi_{\max} \rfloor + 1 + \xi} - \frac{w_B}{\lfloor \xi_{\max} \rfloor + 2 + \xi}$ . It is easy to show that  $\Delta_1 \geq \Delta_2$  for all  $\xi$  values satisfying  $(\lfloor \xi_{\max} \rfloor + 1 + \xi)^2 \geq 2$ , which holds for all the  $\xi_{\max}$  reported in Table 1. Hence we can conclude that the best value for  $a_{Bj}$  is  $\lfloor \xi_{\max} \rfloor + 1$ .

In (11), the right-hand side is maximized when we have the maximum value for  $a_{jW}$  and maximum  $\xi$  or  $\xi_{\max}$ . The left-hand side is minimized for any  $\xi$ , as long as we have  $a_{jW} = \lfloor \xi \rfloor + 1$ . It is clear that  $a_{jW} < \lfloor \xi \rfloor + 1$  makes the left-hand side negative, which makes this lower bound redundant. Considering both sides of (11), the best value for  $\xi$  would be  $\xi_{\max}$ , and having  $\xi = \xi_{\max}$ , the best value for  $a_{jW}$  would be  $a_{jW} \geq \lfloor \xi_{\max} \rfloor + 1$ .

For  $a_{jW} \geq \lfloor \xi_{\max} \rfloor + 1$  and  $\xi = \xi_{\max}$  the upper bound  $w_W(a_{jW} - \xi)$  has the same slop as the lower bound  $w_W(a_{jW} + \xi)$ . This implies that if we move from  $\lfloor \xi_{\max} \rfloor + 1$  to  $\lfloor \xi_{\max} \rfloor + 1 + g$ , for  $g = 1, \dots, a_{BW} - (\lfloor \xi_{\max} \rfloor + 1)$ , the increase in  $w_W(a_{jW} - \xi)$  is equal to the increase in  $w_W(a_{jW} + \xi)$ . As the best value for  $a_{Bj}$  is  $\lfloor \xi_{\max} \rfloor + 1$ ,  $w_W(a_{jW} + \xi)$  becomes the effective upper bound (both  $w_j \leq \frac{w_B}{a_{Bj}-\xi}$  and  $w_j \leq w_B$  become redundant), and then the change in the range is dictated by the lower bound only.  $a_{jW} = \lfloor \xi_{\max} \rfloor + 1$  puts this boundary level  $w_W(a_{jW} - \xi)$  below or on the boundary level  $w_W$ , or equivalently  $w_W(a_{jW} - \xi) \leq w_W$ . This implies that moving from  $a_{jW} = \lfloor \xi_{\max} \rfloor + 1$  to  $a_{jW} = \lfloor \xi_{\max} \rfloor + 2$ , depending on the positioning of the other lower and upper bounds, could widen or shorten the range. This implies that the best value for  $a_{jW}$  is  $\lfloor \xi_{\max} \rfloor + 1$  or  $\lfloor \xi_{\max} \rfloor + 2$ .

Hence, the proof is complete. □

Proposition 1 proves that we need to have  $\xi = \xi_{\max}$  in order to have the widest range. From Rezaei<sup>5</sup> we know that to have the full inconsistency or  $\xi = \xi_{\max}$ , we need to have one criterion (lets call it criterion  $k$ ) with  $a_{Bk} = a_{kW} = a_{BW}$ . This means that

for a problem with  $n$  criteria, the boundaries discussed in Proposition 1 apply to  $n - 3$  criteria (all except for Best, Worst, and  $k$ th).

Based on Proposition 1, we can now find  $\max_{a_{BW},n}(\psi)$  by solving models (5) and (6) for different problems characterized by different  $a_{BW}$  and  $n$  criteria. Assume the 1st criterion as the Best, the  $n$ th criterion as the Worst, and the  $(n - 1)$ th as criterion  $k$ . Considering Proposition 1, we need to find the lower and upper bounds of  $w_j$  for two problems as follows:

Problem 1:

$$\begin{aligned} \text{BO} &= [1, \lfloor \xi_{\max} \rfloor + 1, \dots, \lfloor \xi_{\max} \rfloor + 1, a_{BW}, a_{BW}], \\ \text{OW} &= [a_{BW}, \lfloor \xi_{\max} \rfloor + 1, \dots, \lfloor \xi_{\max} \rfloor + 1, a_{BW}, 1]. \end{aligned}$$

Problem 2:

$$\begin{aligned} \text{BO} &= [1, \lfloor \xi_{\max} \rfloor + 1, \dots, \lfloor \xi_{\max} \rfloor + 1, a_{BW}, a_{BW}], \\ \text{OW} &= [a_{BW}, \lfloor \xi_{\max} \rfloor + 2, \dots, \lfloor \xi_{\max} \rfloor + 2, a_{BW}, 1]. \end{aligned}$$

We then calculate the range for the two problems mentioned above and find  $\max_{a_{BW},n}(\psi)$ .

As can be seen, this requires solving  $2 \times 2n$  LPs of (5) and (6). In the next section, however, we show how the problem can be solved using an analytical approach.

**An analytical approach to find  $\max_{a_{BW},n}(\psi)$**

Based on Proposition 1, here we develop an analytical approach to find  $\max_{a_{BW},n}(\psi)$  as follows.

Let us first establish the relationship between  $w_B$  and  $w_W$ .  $\xi = \xi_{\max}$  characterizes a pairwise comparison system with full inconsistency. We know from Rezaei<sup>5</sup> that for the case of fully inconsistent pairwise comparison system, we have

$$\frac{w_B}{w_W} = a_{BW} + \xi_{\max} \tag{12}$$

or

$$w_W = \frac{w_B}{a_{BW} + \xi_{\max}}. \tag{13}$$

We also know from Rezaei,<sup>5</sup> that full inconsistency happens when for criterion  $k$ , we have  $a_{Bk} = a_{kW} = a_{BW}$ , which means:

$$\frac{w_B}{w_k} = \frac{w_k}{w_W} = a_{BW} - \xi_{\max} \tag{14}$$

or

$$w_k = (a_{BW} - \xi_{\max})w_B. \tag{15}$$

For the remaining  $n - 3$  criteria, we should find the boundaries in Proposition 1. As we know, for each criterion, we have three lower and three upper bounds as follows:

Lower bounds:

$$\frac{w_B}{a_{Bj} + \xi_{\max}}, \quad w_W(a_{jW} - \xi_{\max}), \quad w_W \tag{16}$$

Upper bounds:

$$\frac{w_B}{a_{Bj} - \xi_{\max}}, \quad w_W(a_{jW} + \xi_{\max}), \quad w_B. \tag{17}$$

Replacing  $w_W$  by Eq. (13), we are able to write all the lower and upper bounds (16) and (17) in terms of  $w_B$ . Therefore, we have:

Lower bounds:

$$\frac{w_B}{a_{Bj} + \xi_{\max}}, \quad \frac{w_B}{a_{BW} + \xi_{\max}}(a_{jW} - \xi_{\max}), \quad \frac{w_B}{a_{BW} + \xi_{\max}}. \tag{18}$$

Upper bounds:

$$\frac{w_B}{a_{Bj} - \xi_{\max}}, \quad \frac{w_B}{a_{BW} + \xi_{\max}}(a_{jW} + \xi_{\max}), \quad w_B. \tag{19}$$

We need to find the lower and upper bounds of  $w_j$  for two problems characterized above as Problems 1 and 2.

By replacing the values of BO and OW of Problems 1 and 2 in (18) and (19) we find the lower and upper bounds of  $w_j$  in terms of  $w_B$ . Then by normalizing the coefficients of  $w_j$  we get the weights  $w_j$ . After normalization, the maximum and minimum of each  $w_j$  is found and the range of the weights is easily calculated. The largest ‘sum of the ranges’ of the two problems is  $\max_{a_{BW}, n}(\psi)$ .

In order to illustrate this approach we present an example as follows.

**Example 1.** Let us consider a problem with 6 criteria and  $a_{BW} = 5$ .

Suppose that criterion 1 is the *best* and criterion 6 is the *worst*. For this problem, we should also consider criterion  $k$ , which can be any criterion other than the best and the worst, so let us consider criterion 5.

From Table 1, we see that for a problem with  $a_{BW} = 5$  we have  $\xi_{\max} = 2.3$ .

We should consider two problems as follows:

Problem 1:

$$\begin{aligned} \text{BO} &= [1 \quad 3 \quad 3 \quad 3 \quad 5 \quad 5], \\ \text{OW} &= [5 \quad 3 \quad 3 \quad 3 \quad 5 \quad 1]. \end{aligned}$$

Problem 2:

$$\begin{aligned} \text{BO} &= [1 \quad 3 \quad 3 \quad 3 \quad 5 \quad 5], \\ \text{OW} &= [5 \quad 4 \quad 4 \quad 4 \quad 5 \quad 1]. \end{aligned}$$

Let us start with Problem 1:

We have

$$\frac{w_B}{w_W} = a_{BW} + \xi_{\max},$$

$$\frac{w_B}{w_W} = 5 + 2.3 \quad \text{or} \quad w_W = \frac{1}{7.3} w_B = 0.137 w_B.$$

For the upper bounds  $\frac{w_B}{a_{Bj} - \xi_{\max}}$  and  $w_W(a_{jW} + \xi_{\max})$ , we have:

$$\frac{w_B}{a_{Bj} - \xi_{\max}} = \frac{w_B}{3 - 2.3} = 1.43 w_B,$$

$$w_W(a_{jW} + \xi_{\max}) = 5.3 w_W = 5.3 \frac{w_B}{a_{BW} + \xi_{\max}} = 0.726 w_B.$$

We have another upper bound which is  $w_j \leq w_B$ .

For the lower bounds  $\frac{w_B}{a_{Bj} + \xi_{\max}}$  and  $w_W(a_{jW} - \xi_{\max})$ , we have

$$\frac{w_B}{a_{Bj} + \xi_{\max}} = \frac{w_B}{3 + 2.3} = 0.189 w_B,$$

$$w_W(a_{jW} - \xi_{\max}) = 0.7 w_W.$$

The lower bound  $w_W(a_{jW} - \xi_{\max})$  can be rewritten as

$$w_W(a_{jW} - \xi_{\max}) = 0.7 w_W = 0.7 \frac{w_B}{a_{BW} + \xi_{\max}} = 0.0959 w_B.$$

We also have another lower bound which is  $w_j \geq w_W$ . Again replacing  $w_W$  by (13), we have  $w_j \geq 0.137 w_B$ .

For the minimum of the upper bounds, we have

$$\min\{w_B, 1.43 w_B, 0.726 w_B\} = 0.726 w_B.$$

And for the maximum of the lower bounds, we have

$$\max\{0.137 w_B, 0.189 w_B, 0.0959 w_B\} = 0.189 w_B.$$

For criterion  $k$ , we have

$$w_k = \frac{w_B}{a_{BW} - \xi_{\max}} = \frac{1}{5 - 2.3} w_B = 0.37 w_B.$$

So, now we have all the weights in terms of  $w_B$  as follows (we use superscripts  $l$  and  $u$  for lower bound and upper bound, respectively):

$$w_B = w_1^u = w_1^l = w_1,$$

$$w_2^u = w_3^u = w_4^u = 0.726 w_1,$$

$$w_2^l = w_3^l = w_4^l = 0.189 w_1,$$

$$w_5^u = w_5^l = 0.37 w_1,$$

$$w_W = w_6^u = w_6^l = 0.137 w_1.$$

In order to get the maximum and minimum actual weights of all the criteria, we should consider all the combinations of the criteria with their lower and upper bounds, which are four combinations in this case as follows.

- Case 1.  $[w_1, w_2^u, w_3^u, w_4^u, w_5, w_6] = [1 \ 0.726 \ 0.726 \ 0.726 \ 0.37 \ 0.137]w_1$ .
- Case 2.  $[w_1, w_2^l, w_3^l, w_4^u, w_5, w_6] = [1 \ 0.189 \ 0.726 \ 0.726 \ 0.37 \ 0.137]w_1$
- Case 3.  $[w_1, w_2^l, w_3^l, w_4^l, w_5, w_6] = [1 \ 0.189 \ 0.189 \ 0.726 \ 0.37 \ 0.137]w_1$ .
- Case 4.  $[w_1, w_2^l, w_3^l, w_4^l, w_5, w_6] = [1 \ 0.189 \ 0.189 \ 0.189 \ 0.37 \ 0.137]w_1$ .

By normalizing the coefficients in each vector we get the weights  $w_j$ , for that combination.

- Case 1.  $[w_1, w_2^u, w_3^u, w_4^u, w_5, w_6] = [0.2714 \ 0.1970 \ 0.1970 \ 0.1970 \ 0.1004 \ 0.0372]$ .
- Case 2.  $[w_1, w_2^l, w_3^l, w_4^u, w_5, w_6] = [0.3177 \ 0.0600 \ 0.2306 \ 0.2306 \ 0.1175 \ 0.0435]$ .
- Case 3.  $[w_1, w_2^l, w_3^l, w_4^l, w_5, w_6] = [0.3830 \ 0.0724 \ 0.0724 \ 0.2781 \ 0.1417 \ 0.0525]$ .
- Case 4.  $[w_1, w_2^l, w_3^l, w_4^l, w_5, w_6] = [0.4822 \ 0.0911 \ 0.0911 \ 0.0911 \ 0.1784 \ 0.0661]$ .

Selecting the maximum and minimum for  $w_j$  from the four cases, we have

$$w_1 = [0.2714, 0.4822], \quad w_2 = [0.0600, 0.2781], \quad w_3 = [0.0600, 0.2781],$$

$$w_4 = [0.0600, 0.2781], \quad w_5 = [0.1004, 0.1784], \quad w_6 = [0.0372, 0.0661].$$

We calculate the range which is 0.9717.

We do the same calculations for Problem 2.

Considering  $BO = [1 \ 3 \ 3 \ 3 \ 5 \ 5]$  and  $OW = [5 \ 4 \ 4 \ 4 \ 5 \ 1]$  and repeating similar calculations as we did for Problem 1, we find the range equal to 1.0261, which is greater than 0.9717. So  $\max_{a_{BW}, n}(\psi)$  for  $a_{BW} = 5$  and 6 criteria or  $\max_{a_{BW}=5, n=6}(\psi) = 1.0261$ .

For a better understanding of the problem discussed in the example, see Fig. 1.

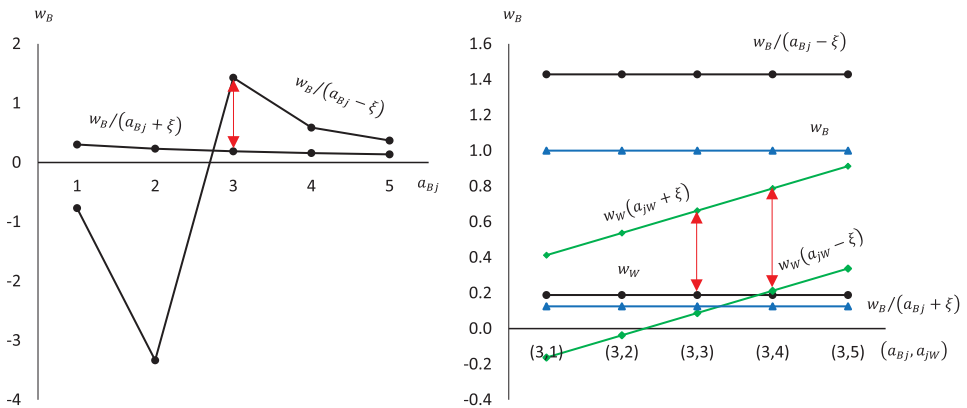


Fig. 1. Visualizing the range of  $w_j$  in terms of  $w_B$  in Example 1.

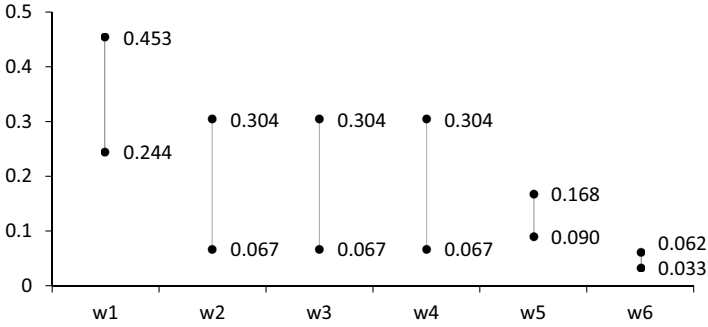


Fig. 2. The intervals of the optimal weights of Problem 2.

The left-hand side of Fig. 1 shows that the maximum range of  $w_j$  in terms of  $w_B$  based on (10) is achieved when  $a_{Bj} = 3$ . By fixing the value of  $a_{Bj} = 3$  in the figure on the right, we see that the maximum range of  $w_j$  in terms of  $w_B$  based on (11) is achieved when  $a_{jW} = 4$ .

Figure 2 shows the intervals of Problem 2, which results in  $\max_{a_{BW}=5, n=6}(\psi) = 1.0261$ .

We do the same calculations for all the other dimensions of the problem (different  $a_{BW}$  and  $n$ ), the results of which are presented in Table 3. To validate the findings, we have also applied models (5) and (6) to get  $\max_{a_{BW}, n}(\psi)$  which leads to the same findings.

As can be seen from Table 3 and Fig. 3, as we increase  $a_{BW}$  from 2 to 9 and the number of criteria  $n$ , from 4 to 9,  $\max_{a_{BW}, n}(\psi)$  increases.

$\kappa$  becomes 1 when the comparison system results in a unique solution, which are all problems with  $n = 2, 3$  or with  $a_{BW} = 1$  or for cases involving more than three criteria, corresponds to a fully consistent pairwise comparison system.  $\kappa$  decreases as  $\max_{a_{BW}, n}(\psi)$  increases, and it reaches to its minimum which is 0. The concentration of the weights is increasing as  $\kappa$  is becoming closer to 1.

Table 3.  $\max_{a_{BW}, n}(\psi)$  for different  $a_{BW}$  and  $n^*$ .

$a_{BW}$	Number of criteria ( $n$ )					
	4	5	6	7	8	9
2	0.1492	0.2524	0.3258	0.3804	0.4219	0.4555
3	0.2828	0.4798	0.6303	0.7486	0.8443	0.9222
4	0.3851	0.6560	0.8607	1.0233	1.1568	1.2677
5	0.4603	0.7815	1.0261	1.2230	1.3860	1.5255
6	0.5200	0.8903	1.1829	1.4274	1.6383	1.8203
7	0.5661	0.9658	1.2829	1.5501	1.7822	1.9870
8	0.5970	1.0157	1.3484	1.6298	1.8758	2.0946
9	0.6178	1.0565	1.4118	1.7180	1.9904	2.2334

Note: \*For  $n = 2, 3$  and  $a_{BW} = 1$ , there is always a unique solution and as a result no range.

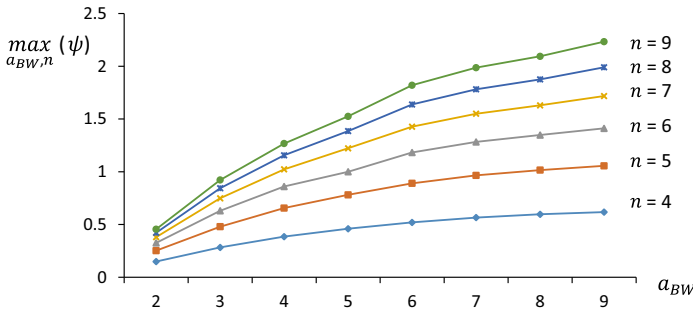


Fig. 3.  $\max_{a_{BW}, n}(\psi)$  for problems with different number of criteria,  $n$ , and different values of  $a_{BW}$ .

### 3.1. Numerical analysis

Assume that we have four decision-making criteria for a particular problem, where criterion 1 is the best and criterion 4 the worst, with the following best-to-others (BO) and others-to-worst (OW) vectors

$$BO = [1 \quad 2 \quad 4 \quad 8],$$

$$OW = [8 \quad 3 \quad 2 \quad 1].$$

Solving the problem using model (3), and then (5) and (6), we have:

$$w_1^* = [0.5455, 0.5576], \quad w_2^* = [0.2355, 0.2407], \quad w_3^* = [0.1292, 0.1481],$$

$$w_4^* = [0.0710, 0.0723] \quad \text{and} \quad \xi^* = 0.3166.$$

The consistency ratio (CR) (see Eq. (4)) of the pairwise comparison system is calculated as follows:

$$CR = \frac{0.3166}{4.47} = 0.071.$$

From Table 3, we have  $\max_{a_{BW}=8, n=4}(\psi) = 0.5970$ . So, the concentration ratio ( $\kappa$ ) can be calculated using Definition 1 as follows:

$$\kappa = 1 - \frac{(0.5576 - 0.5455) + (0.2407 - 0.2355) + (0.1481 - 0.1292) + (0.0723 - 0.0710)}{0.5970} = 0.9372.$$

CR is close to zero and below its associated threshold (see Table 2), or  $0.071 < 0.3409$ .  $\kappa$  is also close to 1, a combination that implies very consistent and concentrated results.

In order to gain greater insight into the relationship between the consistency ratio (CR) and the concentration ratio ( $\kappa$ ), we solved all the instances involving 4 criteria and  $a_{BW} = 5$ . Figure 4 shows the ordered CR and  $\kappa$  of all the instances.

As can be seen from Fig. 4:

- By increasing CR of the comparisons,  $\kappa$  decreases.
- Problems with a common value of CR could have quite different values of  $\kappa$ .

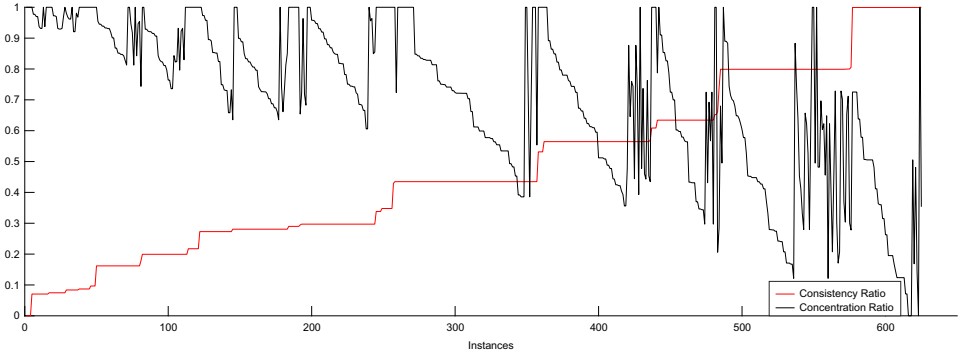


Fig. 4. Consistency ratio and concentration ratio of all instances involving 4 criteria and  $a_{BW} = 5$ .

- Minimum CR or  $CR = 0$  is only associated with maximum  $\kappa$  or  $\kappa = 1$ .
- Maximum  $\kappa$  or  $\kappa = 1$  is associated with different values of CR.
- Minimum  $\kappa$  or  $\kappa = 0$  is only associated with maximum CR or  $CR = 1$ .

The observation strongly suggests the usefulness of the concentration ratio  $\kappa$ , as it enhances our insight about the problem. That is to say, CR tells us about the veracity of the weights and the pairwise comparisons,  $\kappa$  tells us about the concentration of the weights. This confirms that having these two numbers together provides us with greater insight than each of them alone would. More specifically, as we already have thresholds for CR, we do not consider any threshold for  $\kappa$ . When a BWM problem is solved and its CR is below its associated threshold, we will check its concentration ratio to find the flexibility of the weights. From Fig. 4, it appears that for the case of acceptable pairwise comparisons (when the CR does not violate its threshold), we do not have  $\kappa$  close to zero. Such relationship exists for other problems with different dimensions ( $n$ ) and different values of  $a_{BW}$ .

#### 4. Conclusion

The contribution of the paper is the introduction of a ratio that determines the concentration of the results of the nonlinear BWM. The ratio shows the extent to which the optimal weights of the nonlinear BWM tend towards a single point. Investigating the relationship between the concentration ratio and consistency ratio shows that when a pairwise comparison system is fully consistent, it is also concentrated on a single optimal solution. However, when the consistency ratio is increasing, the concentration ratio generally decreases. Nevertheless, problems with the same consistency ratio are characterized with different values of concentration ratio. This implies that the two ratios should be interpreted together. While a problem with a particular consistency ratio could have a concentration ratio less than one, another problem with the same consistency ratio could have a concentration ratio of one. This means that in the former problem, there are some freedom



for the decision-makers to choose an optimal solution from among multiple optimal solutions, while in the latter, the problem has a unique optimal solution. While consistency ratio tests the reliability of the pairwise comparison system provided by the decision-maker(s), the concentration ratio shows the provided flexibility for the decision-makers in choosing the final weights. Future applications of BWM can use the developed concentration ratio next to the consistency ratio to get more insight on the final results of decision-making problems. An interesting future study would be to consider the interactive dialogue with the decision-maker(s) to improve the concentration of the nonlinear BWM. It would be also interesting to have a deeper look at the relationship between the consistency ratio and the concentration ratio when the consistency ratio satisfies the threshold. Finally, we think that the idea proposed in this study can be used to devise similar consternation ratios for other variants of the BWM such as the multiplicative BWM<sup>35</sup> and the Bayesian BWM<sup>36</sup> as well as for other methods which result in multiple optimal solutions including UTA family.<sup>37</sup>

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